Six-vertex model with domain wall boundary conditions and one-matrix model

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The partition function of the six-vertex model on a square lattice with domain wall boundary conditions (DWBC) is rewritten as a Hermitian one-matrix model or a discretized version of it (similar to sums over Young diagrams), depending on the phase. The expression is exact for finite lattice size, which is equal to the size of the corresponding matrix. In the thermodynamic limit, the matrix integral is computed using traditional matrix model techniques, thus providing a complete treatment of the bulk free energy of the six-vertex model with DWBC in the different phases. In particular, in the antiferroelectric phase, the bulk free energy and a subdominant correction are given exactly in terms of elliptic theta functions.

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I. INTRODUCTION

In [1], Korepin and the author brought up the issue of the sensitivity of the six-vertex model to its boundary conditions (even in the thermodynamic limit). The motivation came mostly from some recent work on domino tilings [2,3,4], in which boundary conditions seemed to affect greatly the typical arrangement of dominos. The problem of counting domino tilings is equivalent to the six-vertex model with particular Boltzmann weights; this is schematically described on Fig. 1. Therefore it seems natural to investigate the corresponding problem for the general six-vertex model with arbitrary weights.

The usual studies of the six-vertex model (see [5] and references therein) are made by assuming periodic boundary conditions (PBC). In [1], different boundary conditions, the so-called domain wall boundary conditions (DWBC), were used [Fig. 2(a)], and the thermodynamic limit of the model was investigated using determinant formulas for the partition function [6,7]. The main result found was an expression for the bulk free energy in the disordered phase of the model, which is different from the usual expression for the case of periodic boundary conditions. It should be noted that the DWBC correspond to the Aztec shape in the domino tiling language (see Fig. 2), which is precisely the type of tiling which was considered in [2,3].

Here, we use a new method to compute the bulk free energy with DWBC in all phases of the model; in particular, we obtain an independent confirmation of the results of [1]. In Sec. II, starting from the determinant formula for the partition function, we shall rewrite the latter as a matrix integral, but with a measure on the space of hermitean matrices which is not necessarily smooth. In the disordered phase (Sec. IV), the measure will turn out to be smooth, whereas in the ferroelectric and antiferroelectric phase (Secs. III and V) it will be discrete (when expressed in terms of the eigenvalues). The size N of the matrices is the size of the original square lattice, and therefore the thermodynamic limit can be investigated using tools from large N matrix models. Since the results of Sec. V (concerning the antiferroelectric phase) are new, they are analyzed in more detail by considering various limits of the parameters, and the subleading correction of the free energy is calculated.

II. PROPERTIES OF THE DETERMINANT FORMULA

We use the same notations as in [1]. We consider the homogeneous six-vertex model, with the following parametrization of the Boltzmann weights attached to the vertices:

$$a=\sinh(t-\gamma),\quad b=\sinh(t+\gamma),\quad c=\sinh(2\gamma). \tag{2.1}$$

The domain wall boundary conditions (DWBC) mean that external horizontal arrows are outgoing, whereas external vertical arrows are incoming [Fig. 2(a)]. These boundary conditions only exist for square lattices. In [6,7], it was shown that the partition function of the six-vertex model with DWBC on a $N\times N$ lattice could be written as

$$Z_N=\frac{[\sinh(t+\gamma)\sinh(t-\gamma)]^{N^2}}{(\Pi_{n=0}^{N-1}n!)^2}\,\tau_N, \tag{2.2}$$

where τ_N is a Hänkel determinant:

$$\tau_N=\det_{1\leqslant i,k\leqslant N}\left[\frac{d^{i+k-2}}{dt^{i+k-2}}\,\phi(t)\right]. \tag{2.3}$$

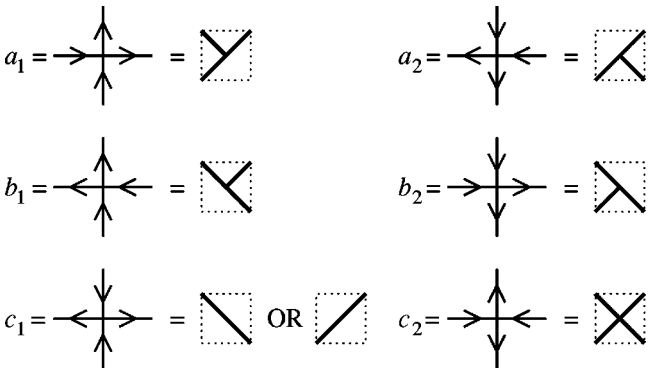

FIG. 1. Correspondence between vertices of the six-vertex model and small patches of a domino tiling.

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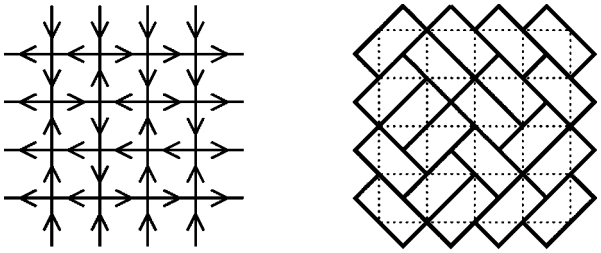

FIG. 2. (a) A configuration of the six-vertex model with DWBC, and (b) one possible corresponding tiling of the Aztec diamond.

Here,

$$\phi(t)\equiv\frac{\sinh(2\gamma)}{\sinh(t+\gamma)\sinh(t-\gamma)}. \tag{2.4}$$

It is known that such determinants are tau functions of the Toda semi-infinite chain hierarchy in terms of appropriate parameters. Here, as a function of t, the τ_N satisfy the usual Toda equations under the bilinear form [8,1]:

$$\tau_N\tau_N''-\tau_N'^2=\tau_{N+1}\tau_{N-1},\quad \forall N\geqslant 1. \tag{2.5}$$

This equation was used in [1] to derive the bulk free energy of the model in the ferroelectric and disordered phase by making an appropriate Ansatz on the large N form of τ_N. Unfortunately, the Ansatz in the antiferroelectric phase is not that simple, as we shall see, and would be hard to justify at this point.

We shall therefore use another approach here, based on the equivalence of Hänkel determinants with one-matrix models [9,10]. Let us write formally $\phi(t)$ as a Laplace transform:

$$\phi(t)=\int dm(\lambda)e^{t\lambda}, \tag{2.6}$$

where $dm(\lambda)$ is a measure. We then notice that the derivatives of $\phi(t)$ are the moments:

$$\frac{d^i}{dt^i}\phi(t)=\int dm(\lambda)\lambda^i e^{t\lambda}. \tag{2.7}$$

Inserting this into Eq. (2.3) leads to

$$\tau_N=\int dm(\lambda_1)\cdots dm(\lambda_N)\sum_{\sigma\in S_N}(-1)^\sigma\prod_{i=1}^N[e^{t\lambda_i}\lambda_i^{i+\sigma(i)-2}]. \tag{2.8}$$

We see that appears naturally the Van der Monde determinant $\Delta(\lambda_i)=\det(\lambda_i^{j-1})=\Pi_{i<j}(\lambda_i-\lambda_j)$. After a few elementary manipulations we find:

$$\tau_N=\frac{1}{N!}\int dm(\lambda_1)\cdots dm(\lambda_N)\Delta(\lambda_i)^2e^{t\Sigma_i\lambda_i}. \tag{2.9}$$

If $dm(\lambda)$ is a smooth positive measure of the form $dm(\lambda)=d\lambda e^{-V(\lambda)}$, then we recognize in Eq. (2.9) the expression in terms of its eigenvalues of the matrix integral:

$$\tau_N\sim\int dM e^{\mathrm{tr}[tM+V(M)]}, \tag{2.10}$$

where M is a Hermitian $N\times N$ matrix, and dM is the flat measure.

As we shall see, if the measure is not smooth, we shall end up with expressions which can still be treated using appropriately adapted matrix model techniques. This is typically the case of discrete measures that appear in sums over Young diagrams [11–18].

Expressions of the type (2.9) have been widely studied in the literature (on random matrices in particular). One important goal is to find their large N asymptotic behavior. Here we shall mention the simplest method to find their leading large N behavior: the saddle point method. The basic idea is that $\log\Delta(\lambda_i)^2$, being a sum of $\sim N^2$ terms, scales as N^2 in the large limit, whereas there are only N variables of integration. Therefore the integral is dominated by a saddle point. An important remark is that, in order to find the saddle point, we must write our action (i.e., log of the function integrated) in such a way that all terms are of the same order N^2. Here, the term $t\Sigma_i\lambda_i$ is naively of order N, and we reach the important conclusion that the λ_i will scale as

$$\lambda_i\propto N\mu_i. \tag{2.11}$$

After the change of variables $\lambda_i\to\mu_i$, one can use the saddle point approximation, which gives us access to the function f defined by

$$f=\lim_{N\to\infty}\frac{\log(\tau_N/c_N)}{N^2}, \tag{2.12}$$

where $c_N\equiv(\Pi_{n=0}^{N^2}n!)^2$. f is essentially the bulk free energy, cf. Eq. (2.2). Note that the saddle point is a very crude approximation in the sense that it does not naturally allow for a systematic computation of subleading corrections; however it will be sufficient for our purposes. We now proceed with a separate discussion of the different phases of the model.

III. FERROELECTRIC PHASE

This is the phase in which the weights are given by Eq. (2.1) with t and γ real, $|\gamma|<t$. We use the following decomposition:

$$\phi(t)=\frac{\sinh(2\gamma)}{\sinh(t+\gamma)\sinh(t-\gamma)}=4\sum_{l=0}^{\infty}e^{-2tl}\sinh(2\gamma l). \tag{3.1}$$

We are in the situation where the measure dm is discrete. The determinant takes the form

$$\tau_N=2^{N^2}\sum_{l_1,\ldots,l_N=0}^{\infty}\Delta(l_i)^2e^{-2t\Sigma_i l_i}\prod_i\sinh(2\gamma l_i) \tag{3.2}$$

(we have neglected here, as in all subsequent calculations, constant factors which manifestly do not contribute to the bulk free energy). This expression is very close to what one encounters when studying the Plancherel measure (or other similar measures) on Young diagrams [11]. In the context of

Young diagrams, the l_i represent the shifted highest weights $l_i = m_i + N - i$, where the m_i are the usual highest weights (sizes of the rows of the diagram), and one is usually interested in the limiting shape of the Young diagram when its size is sent to infinity. There has been a lot of work on this type of expressions, both in the mathematical literature [11,15–17] (the recent work being concerned with *fluctuations* around the limiting shape, which we shall not discuss here) and the physical literature [12–14,18]. One relevant observation from [12] is the following: after the rescaling $\mu = l/N$, all sums look like Rieman sums and one is tempted to replace them with integrals, and then apply the saddle point method. This is correct *on condition* that one imposes an additional constraint coming from the discreteness of the l_i. In Eq. (3.2), all l_i must be distinct integers (due to the Van der Monde determinant), and therefore

$$|l_i - l_j| \geqslant 1, \quad \forall i \neq j. \tag{3.3}$$

If we introduce the density $\rho(\mu)d\mu$ of the $\mu_i = l_i/N$, normalized so that $\int \rho(\mu)d\mu = 1$, then Eq. (3.3) implies that it must satisfy the inequality

$$\rho(\mu) \leqslant 1. \tag{3.4}$$

In general, when the l_i are trapped in a well of the potential (as is the case here), there will be a saturated region at the bottom of the well where $\rho(\mu) = 1$, and an unsaturated region where $\rho(\mu) < 1$.

Let us now proceed with the solution. Once the rescaling $\mu_i = l_i/N$ is performed, one notices that up to corrections exponentionally small in N, $\sinh(2\gamma N \mu_i) \approx 1/2 e^{2|\gamma| N \mu_i}$. Therefore

$$\tau_N \approx c_N' 2^{N^2} \sum_{\mu_1, \ldots, \mu_N \in (1/N)\mathbb{Z}_+} \Delta(\mu_i)^2 e^{-2N(t-|\gamma|)\Sigma_i \mu_i}, \tag{3.5}$$

where $c_N' \equiv N^{N^2}$. Of course, once this simplification is made, we regognize a well-known expression; in fact, going back now to the original variables l_i one can compute τ_N directly using the Cauchy identity for Schur functions. However, to emphasize the similarity with the other phases (which do not possess such a simple group-theoretic interpretation), we shall use the saddle point method, following the solution of [18]. Since τ_N only depends on $t - |\gamma|$, we temporarily set $\gamma = 0$.

The support of the saddle point density $\rho(\mu)$ is expected to be of the form $[0, \beta]$; the saturated region is $[0, \alpha]$, whereas the unsaturated region is $[\alpha, \beta]$. We define the resolvent

$$\omega(z) = \int_0^\beta \frac{d\mu \rho(\mu)}{z - \mu} \tag{3.6}$$

for all complex $z \notin [0, \beta]$. The saddle point equations can be written in terms of ω:

$$\omega(\mu + i0) + \omega(\mu - i0) = 2t, \quad \forall \mu \in [\alpha, \beta]. \tag{3.7}$$

In order to solve the equation, we first remove the logarithmic cut of ω with the redefinition: $\tilde{\omega}(z) = \omega(z) - \log[\mu/(\mu - \alpha)]$ $\tilde{\omega}(z)$ is analytic everywhere except on $[\alpha, \beta]$ and satisfies

$$\tilde{\omega}(\mu + i0) + \tilde{\omega}(\mu - i0) = 2t - 2\log\frac{\mu}{\mu - \alpha}. \tag{3.8}$$

This completely determines it to be

$$\tilde{\omega}(z) = t - \sqrt{(z-\alpha)(z-\beta)} \times \int_{\alpha - i0}^{\beta - i0} \frac{dz'}{2i\pi(z - z')\sqrt{(z'-\alpha)(z'-\beta)}} \log\frac{z'}{z' - \alpha}. \tag{3.9}$$

After some calculations, we find that

$$\omega(z) = t - 2\log\left[\frac{\sqrt{\beta(z-\alpha)} + \sqrt{\alpha(z-\beta)}}{\sqrt{z(\beta - \alpha)}}\right]. \tag{3.10}$$

The end points α and β are determined by imposing $\omega(z) \sim 1/z$ as $z \to \infty$. This gives rise to two equations:

$$t = \log\frac{\sqrt{\beta} + \sqrt{\alpha}}{\sqrt{\beta} - \sqrt{\alpha}},$$

$$\sqrt{\alpha\beta} = 1, \tag{3.11}$$

whose solution is

$$\alpha = \coth\frac{t}{2}, \quad \beta = \tanh\frac{t}{2}. \tag{3.12}$$

In order to conclude, one expands further the function $\omega(z)$:

$$\omega(z) = \frac{1}{z} + \frac{\alpha + \beta}{4}\frac{1}{z^2} + \cdots \tag{3.13}$$

and uses the fact that

$$\frac{\partial f}{\partial t} = -2\langle \mu \rangle = -\frac{\alpha + \beta}{2} = \coth t. \tag{3.14}$$

Integrating once and restoring γ, we have the final result

$$e^f = \frac{1}{\sinh(t - |\gamma|)}, \tag{3.15}$$

which coincides with what was found in [1].

IV. DISORDERED PHASE

In this phase, one usually rewrites the weights

$$a = \sin(\gamma - t), \quad b = \sin(\gamma + t), \quad c = \sin(2\gamma), \tag{4.1}$$

with redefined parameters t and γ, $|t| < \gamma$, and the function $\phi(t) = \sin(2\gamma)/[\sin(t - \gamma)\sin(t + \gamma)]$; the partition function is then given by

$$Z_N=\frac{[\sin(\gamma+t)\sin(\gamma-t)]^{N^2}}{(\Pi_{n=0}^{N-1}n!)^2}\tau_N, \tag{4.2}$$

with τ_N still given by Eq. (2.3). The Laplace transform is

$$\phi(t)=\frac{\sin(2\gamma)}{\sin(\gamma+t)\sin(\gamma-t)}=\int_{-\infty}^{+\infty}d\lambda e^{t\lambda}\frac{\sinh\frac{\lambda}{2}(\pi-2\gamma)}{\sinh\frac{\lambda}{2}\pi}. \tag{4.3}$$

This time the measure is smooth and τ_N is a matrix integral in the usual sense.

We must now rescale the variables λ_i. We choose to define $\mu_i=\gamma\lambda_i/N$. Then

$$\tau_N=c_N'\gamma^{-N^2}\int_{-\infty}^{+\infty}d\mu_1\cdots d\mu_N\Delta(\mu_i)^2 \times\prod_{i=1}^{N}\left[\frac{\sinh N\mu_i\left(\frac{\pi}{2\gamma}-1\right)}{\sinh N\mu_i\frac{\pi}{2\gamma}}e^{N(t/\gamma)\mu_i}\right]. \tag{4.4}$$

One then simplifies the potential by using $\sinh N\mu(\pi/2\gamma-1)/\sinh N\mu(\pi/2\gamma)\sim e^{-N|\mu|}$. Therefore,

$$\tau_N\approx c_N'\gamma^{-N^2}\int_{-\infty}^{+\infty}d\mu_1\cdots d\mu_N\Delta(\mu_i)^2e^{N\Sigma_i[(t/\gamma)\mu_i-|\mu_i|]}. \tag{4.5}$$

Note that the matrix integral only depends on the ratio $\zeta\equiv t/\gamma$.

The matrix model (4.5) is fairly simple and can be solved easily in the large N limit via the saddle point method. One introduces again the saddle point density of eigenvalues $\rho(\mu)d\mu$, normalized so that $\int\rho(\mu)d\mu=1$. The support of $\rho(\mu)$ is assumed to be a single interval $[\alpha,\beta]$ ($\alpha<0<\beta$), due to the shape of the potential (single well centered around 0). The resolvent is defined as before. The saddle point equations read

$$\omega(\mu+i0)+\omega(\mu-i0)=-\zeta+\mathrm{sgn}(\mu),\quad \forall\mu\in[\alpha,\beta], \tag{4.6}$$

where the right-hand side is simply the derivative of the potential. The solution of this equation

$$\omega(z)=\frac{1-\zeta}{2}+\frac{2}{i\pi}\log\left[\frac{\sqrt{\beta(z-\alpha)}-i\sqrt{-\alpha(z-\beta)}}{\sqrt{z(\beta-\alpha)}}\right] \tag{4.7}$$

is very similar to the ferroelectric phase; and the rest of the calculation goes along the same lines.

Requiring that $\omega(z)\sim 1/z$ as $z\to\infty$, we obtain the two equations:

$$1-\zeta=\frac{2}{i\pi}\log\frac{\sqrt{\beta}+i\sqrt{-\alpha}}{\sqrt{\beta}-i\sqrt{-\alpha}},$$

$$\sqrt{-\alpha\beta}=\pi, \tag{4.8}$$

which we solve for α and β:

$$\alpha=-\pi\tan\frac{\pi}{4}(1-\zeta),\quad \beta=\pi\tan\frac{\pi}{4}(1+\zeta). \tag{4.9}$$

Noting that

$$\frac{\partial f}{\partial\zeta}=\left\langle\frac{1}{N}\mathrm{tr}M\right\rangle=\frac{\alpha+\beta}{4}, \tag{4.10}$$

we find

$$f=-\log\cos\frac{\pi}{2}\zeta+\text{const.} \tag{4.11}$$

We shall not discuss how to fix the constant of integration, since this will be addressed in the next section in a more general setting. Reintroducing the γ dependence coming from Eq. (4.5), we have the final expression:

$$e^f=\frac{\pi}{2\gamma}\frac{1}{\cos\frac{\pi t}{2\gamma}}, \tag{4.12}$$

which reproduces the result of [1].

V. ANTIFERROELECTRIC PHASE

We finally study the most interesting phase, in which the weights are given by

$$a=\sinh(\gamma-t),\quad b=\sinh(\gamma+t),\quad c=\sinh(2\gamma), \tag{5.1}$$

with $|t|<\gamma$, and the partition function by

$$Z_N=\frac{[\sinh(\gamma+t)\sinh(\gamma-t)]^{N^2}}{(\Pi_{n=0}^{N-1}n!)^2}\tau_N, \tag{5.2}$$

with $\phi(t)=\sinh(2\gamma)/[\sinh(\gamma+t)\sinh(\gamma-t)]$.

A. Bulk free energy

We have the expansion

$$\phi(t)=\frac{\sinh(2\gamma)}{\sinh(\gamma+t)\sinh(\gamma-t)}=2\sum_{l=-\infty}^{+\infty}e^{2tl}e^{-2\gamma|l|}. \tag{5.3}$$

We perform the rescaling $\mu_i=2\gamma l_i/N$ and find that τ_N takes the form

$$\tau_N=c_N'\gamma^{-N^2}\sum_{\mu_1,\ldots,\mu_N\in(2\gamma/N)\mathbb{Z}}\Delta(\mu_i)^2e^{n\Sigma_i[(t/\gamma)\mu_i-|\mu_i|]}. \tag{5.4}$$

The remarkable feature is that Eq. (5.4) is identical to Eq. (4.5) up to the discrete nature of the variables. We shall comment on this later.

The situation is a bit more complicated than in the previous cases, since we now expect a saturated region $[\alpha',\beta']$ at

the bottom of the well ($\alpha'<0<\beta'$) and *two* unsaturated regions $[\alpha,\alpha']$ and $[\beta',\beta]$ on each side. This is a two-cut situation, which is in fact the reason why the naive approach of [1] fails in the antiferroelectric phase (see Sec. V D for more on this). Let us define as before $\zeta=t/\gamma$, the density $\rho(\mu)$ and its resolvent $\omega(\mu)$. The constraint coming from the discreteness of the μ_i reads

$$\rho(\mu)\leqslant\frac{1}{2\gamma},\quad \forall\mu. \tag{5.5}$$

Therefore we have in the saturated region the equation

$$\rho(\mu)=\frac{1}{2i\pi}[\omega(\mu-i0)-\omega(\mu+i0)]=\frac{1}{2\gamma},$$

$$\forall\mu\in[\alpha',\beta'], \tag{5.6}$$

whereas in the unsaturated regions, the saddle point equations are

$$\omega(\mu+i0)+\omega(\mu-i0)=-\zeta+\mathrm{sgn}(\mu),$$

$$\forall\mu\in[\alpha,\alpha']\cup[\beta',\beta], \tag{5.7}$$

with $\zeta=t/\gamma$.

We could proceed as in the previous sections; this would lead to a representation of $\omega(z)$ in terms of elliptic integrals. However, this would be fairly cumbersome and we proceed instead as follows. Introduce an elliptic parametrization

$$u(\mu)=\frac{1}{2}\sqrt{(\beta'-\alpha)(\beta-\alpha')}\times\int_\beta^\mu\frac{dz}{\sqrt{(z-\alpha)(z-\alpha')(z-\beta')(z-\beta)}}, \tag{5.8}$$

which corresponds to setting:

$$[(\beta'-\alpha)/(\beta-\alpha)][(\beta-\mu)/(\beta'-\mu)]=\mathrm{sn}^2(u,k)$$

with $k=\sqrt{(\beta-\alpha)(\beta'-\alpha')/(\beta'-\alpha)(\beta-\alpha')}$. With an appropriate choice of path of integration, this maps the μ complex plane (respectively upper half-plane, lower half-plane) onto the rectangle $[0,K]\times[-iK',iK']$ (respectively $[0,K]\times[0,iK']$,$[0,K]\times[-iK',0]$), where K and K' are the usual complete elliptic integrals of the first kind. Similarly, the second sheet of the double covering is mapped onto the other half of the torus, which can be chosen to be $[-K,0]\times[-iK',iK']$. The point of this parametrization is that the resolvent ω is now a well-defined function of u. In fact we have the following properties.

(i) The function $\omega(u)$ can be extended to a homomorphic function in the whole u plane.

(ii) The function $\omega(u)$ satisfies the following functional relations (for all complex u):

$$\omega(u+2iK')=\omega(u)-\frac{i\pi}{\gamma}, \tag{5.9a}$$

$$\omega(u+2K)=\omega(u)-2, \tag{5.9b}$$

$$\omega(u)+\omega(-u)=1-\zeta. \tag{5.9c}$$

Equation (5.9a) is the analytic continuation of Eq. (5.6). Similarly, by combining the analytic continuations of the two equations contained in Eq. (5.7), one obtains Eqs. (5.9b) and (5.9c).

(iii) The function $\omega(u)$ has the following expansion near $u_\infty=u(z=\infty)$:

$$\omega=-\frac{2}{\sqrt{(\beta'-\alpha)(\beta-\alpha')}}(u-u_\infty)+O(u-u_\infty)^2. \tag{5.10}$$

This is a rewriting of the condition $\omega(z)\sim 1/z$ at infinity.

Using properties (i) and (ii) [Eqs. (5.9a) and (5.9b)], we conclude that $(d/du)\omega(u)$ is a doubly periodic holomorphic function, and so is a constant. In order to restore the coefficients of $\omega(u)$ we can use properties (ii) or (iii). We find that

$$\omega(u)=-\frac{1}{K}(u-u_\infty) \tag{5.11}$$

plus several conditions relating the different parameters of the problem:

$$\frac{K'}{K}=\frac{\pi}{2\gamma}, \tag{5.12a}$$

$$\sqrt{(\beta'-\alpha)(\beta-\alpha')}=2K, \tag{5.12b}$$

$$\frac{u_\infty}{K}=\frac{1-\zeta}{2}. \tag{5.12c}$$

Relation (5.12a) is particularly interesting since it shows that the elliptic nome $q=e^{-\pi K'/K}=e^{-\pi^2/2\gamma}$ depends only on γ (and not on ζ). Also, the dual nome (under modular transformation) $\tilde{q}=e^{-2\gamma}$ is up to a sign the quantum group deformation parameter of the model.

We can rewrite the three conditions in terms of the end points; we find

$$\beta-\alpha=2K\frac{\mathrm{dn}\,u_\infty}{\mathrm{sn}\,u_\infty\mathrm{cn}\,u_\infty},$$

$$\beta-\alpha'=2K\frac{\mathrm{cn}\,u_\infty}{\mathrm{sn}\,u_\infty\mathrm{dn}\,u_\infty}, \tag{5.13}$$

$$\beta-\beta'=2K\frac{\mathrm{cn}\,u_\infty\mathrm{dn}\,u_\infty}{\mathrm{sn}\,u_\infty}.$$

In order to completely fix the four end points α, α', β', β, we need one extra relation; this is the equality of chemical potentials in the two unsaturated regions. This relation takes the form

$$\int_{\alpha'}^{\beta'}[\omega(\mu+i0)+\omega(\mu-i0)]d\mu=(1-\zeta)\beta'+(1+\zeta)\alpha'. \tag{5.14}$$

Using the expression (5.11) of $\omega(u)$, we can rewrite it as

$$\beta'-(\beta-\beta')\frac{\mathrm{sn}\,u_\infty}{\mathrm{cn}\,u_\infty\mathrm{dn}\,u_\infty}Z(u_\infty)=0, \tag{5.15}$$

where Z is Jacobi's zeta function; this fixes b' to be

$$\beta' = 2KZ(u_\infty). \tag{5.16}$$

The end points are now determined by Eqs. (5.13) and (5.16), supplemented by the value (5.12c) of u_∞.

At this point, we are ready to calculate the free energy. We first rewrite explicitly the resolvent [Eq. (5.11)] under the form

$$\omega(z) = \int_z^\infty \frac{dz'}{\sqrt{(z'-\alpha)(z'-\alpha')(z'-\beta')(z'-\beta)}}. \tag{5.17}$$

Next we expand it to order $1/z^2$ to find

$$\frac{\partial f}{\partial \zeta} = \frac{\alpha+\alpha'+\beta'+\beta}{4}, \tag{5.18}$$

which generalizes Eq. (4.10); using some known identities satisfied by zeta and theta functions, we obtain

$$\frac{\partial f}{\partial \zeta} = -\frac{\pi}{2}\frac{\theta_2'(\pi\zeta/2)}{\theta_2(\pi\zeta/2)}, \tag{5.19}$$

where we recall that $\theta_2(z)$ is

$$\theta_2(z) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos(2n+1)z. \tag{5.20}$$

There are a variety of ways to find the integration constant. One is to calculate explicitly f (for a particular value of ζ, e.g., $\zeta=0$) using this matrix model solution, and then restore the γ dependence coming from Eq. (5.4); this is a straightforward but tedious exercise. Another possibility is to use the known limits $\zeta \to \pm 1$, that is $t \to \pm\gamma$, where we should have (see [1])

$$e^f \sim \frac{1}{\gamma \mp t}. \tag{5.21}$$

Either way, we finally find

$$e^f = \frac{\pi}{2\gamma}\frac{\theta_1'(0)}{\theta_2\left(\dfrac{\pi t}{2\gamma}\right)}, \tag{5.22}$$

where we recall that the elliptic nome is $q = e^{-\pi^2/2\gamma}$.

As a simple check of our calculation, note that if one sends γ to 0 (keeping ζ fixed), since the constraint (5.5), which was the only difference with the disordered phase, disappears, one should recover the results of the previous section. This is indeed what happens when one replaces the theta functions with their $q \to 0$ limit. Also, Eq. (5.22) has been numerically checked with high accuracy.

This concludes the calculation of the bulk free energy in the anti-ferroelectric phase. Restated more explicitly, this is the result we have obtained: the partition function Z_N of the six-vertex model on a $N \times N$ lattice with DWBC and Boltzmann weights given by Eq. (5.1) has the following large N behavior:

$$\lim_{N\to\infty} Z_N^{1/N^2} = \sinh\gamma(1-\zeta)\sinh\gamma(1+\zeta)\frac{\pi}{2\gamma}\frac{\theta_1'(0)}{\theta_2\left(\dfrac{\pi\zeta}{2}\right)}, \tag{5.23}$$

where $\zeta = t/\gamma$, and the elliptic nome of the theta functions is $q = e^{-\pi^2/2\gamma}$. Note that this expression is different from the corresponding expression for PBC. Let us now consider the two limits $\gamma \to 0$ and $\gamma \to \infty$. In both cases we shall assume that ζ remains fixed.

B. Small γ limit

As one sends γ to 0, one reaches the line of the disordered/antiferroelectric phase transition. As noted earlier, the bulk free energy of the disordered phase is essentially obtained from that of the antiferroelectric phase by setting $q=0$ in the theta functions (and performing the rotation $\gamma \to i\gamma, t \to it$ in the prefactors). Considering that $q = e^{-\pi^2/2\gamma}$, we expect a very smooth phase transition. More explicitly, we have the following expansion of f:

$$f = \log\left[\frac{\pi}{2\gamma}\frac{1}{\cos\left(\dfrac{\pi t}{2\gamma}\right)}\right] + 2 \times \sum_{m=1}^{\infty}\frac{1}{m}\frac{q^{2m}}{1-q^{2m}}(1-(-1)^m\cos[m\pi t/\gamma)]. \tag{5.24}$$

After substraction of the analytic continuation of the disordered phase free energy (note that this analytic continuation is trivial since f only depends on t/γ), we obtain the singular part of the free energy, which has a leading singularity

$$f_{\text{sing}} = 4e^{-\pi^2/\gamma}\cos^2\left(\frac{\pi t}{2\gamma}\right) + \cdots. \tag{5.25}$$

This is the same type of singularity that appears in the model with periodic boundary conditions [5]. In more physical terms, if we introduce a temperature T which is near the critical temperature T_c, we have

$$f_{\text{sing}} \propto e^{-C/\sqrt{T_c - T}}, \tag{5.26}$$

that is an infinite order phase transition.

C. Large γ limit

Next, let us consider the $\gamma \to \infty$ limit, i.e., $\Delta = -\cosh(2\gamma) \to -\infty$. This is a typical zero temperature limit, and we expect that the free energy will be dominated by the contribution of a ground state. After a modular transformation, the bulk free energy reads

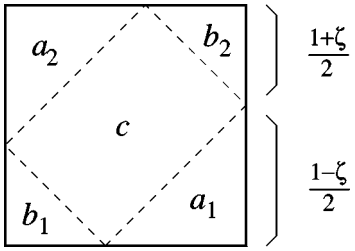

FIG. 3. Ground states of the antiferroelectric phase. In regions *a* and *b* the arrows are aligned, whereas in region *c* they alternate in direction.

$$F = -\log[\sinh(\gamma - t)\sinh(\gamma + t)] - f$$
$$= -\frac{\gamma}{2} - \frac{t^2}{2\gamma} - \log\sinh(\gamma + t) + t$$
$$+ 2\sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{-2m\gamma}}{\sinh(2m\gamma)} \sinh^2[m(\gamma - t)]. \quad (5.27)$$

We can interpret the first terms when $\gamma \to \infty$

$$F = -\frac{3}{2}\gamma - \frac{t^2}{2\gamma} + O(e^{-2\gamma}) \quad (5.28)$$

as coming from the family of ground states described by Fig. 3. The pattern of a rectangle inscribed inside a square is reminiscent of the circle inscribed inside a square characteristic of the disordered phase [2].

D. Subdominant corrections

As a final note, it is interesting to understand why the approach of [1] fails in the antiferroelectric phase. There, the idea was to find an appropriate ansatz on the asymptotic behavior of the determinant τ_N and plug it in the Toda equation (2.5). The simplest assumption is that only the leading behavior (bulk free energy) must be taken into account, which leads to replacing τ_N with $c_N e^{N^2 f}$, where $c_N = (\Pi_{n=0}^{N-1} n!)^2$. The Toda equation then reduces to the ordinary differential equation for f:

$$f'' = e^{2f}. \quad (5.29)$$

We can now use some insight from matrix models to understand whether this assumption was justified or not. In the ferroelectric and disordered phases, we reduced the computation of τ_N to a matrix model with eigenvalues in *one single interval* $[a,b]$ (disregarding the saturated region which plays no role here); it is known that such models have a regular large N limit. In fact, in the ferroelectric phase one can easily prove that

$$\tau_N \sim c_N e^{N^2 f} e^{N(t - |\gamma|)} \quad (5.30)$$

up to only exponentially small corrections; whereas in the disordered phase, one expects an asymptotic expansion which starts with

$$\tau_N \sim c_N e^{N^2 f} N^{\kappa} C \quad (5.31)$$

and continues with inverse powers of N (note that this is not quite the usual topological expansion of 2D gravity since the potential is not polynomial). In either case, the assumption on the corrections is valid, and indeed, one can check that the expressions (3.15) and (4.12) do satisfy the ODE (5.29).

On the contrary, in the antiferroelectric regime, we have found that the support of the eigenvalues contains *two intervals* $[a,a']$ and $[b',b]$ and therefore we expect to be in a situation similar to what was studied in [19,20]. The analysis shows that τ_N should in this case display a pseudoperiodic behavior, which is indeed what is found in numerical computations. More precisely, after some calculations along the lines of [20], one finds that

$$\tau_N \sim c_N \left[\frac{\pi}{2\gamma} \frac{\theta_1'(0)}{\theta_2\left(\frac{\pi\zeta}{2}\right)} \right]^{N^2} \theta_4\left(\frac{\pi}{2}(1+\zeta)N\right) C, \quad (5.32)$$

where $\zeta = t/\gamma$ and the elliptic nome q of the theta function is as before $q = e^{-\pi^2/2\gamma}$. The constant C depends only on γ. One can check that the right-hand side of Eq. (5.32) does satisfy the Toda equation (2.5), even though the bulk free energy alone does not satisfy the ODE (5.29).

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